CERN-PH-TH/2012-143

Invariant Differential Operators for Non-Compact Lie Groups: the $Sp(n, \mathbb{R})$ Case[1]

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Abstract

In the present paper we continue the project of systematic construction of invariant differential operators on the example of the non-compact algebras $sp(n, \mathbb{R})$, in detail for $n = 6$. Our choice of these algebras is motivated by the fact that they belong to a narrow class of algebras, which we call 'conformal Lie algebras', which have very similar properties to the conformal algebras of Minkowski space-time. We give the main multiplets and the main reduced multiplets of indecomposable elementary representations for $n = 6$, including the necessary data for all relevant invariant differential operators. In fact, this gives by reduction also the cases for $n < 6$, since the main multiplet for fixed n coincides with one reduced case for $n + 1$.

[1]To appear in the Proceedings of the 9. International Workshop *Lie Theory and Its Applications in Physics* (LT-9), Varna, Bulgaria, June 2011.

1 Introduction

Consider a Lie group G, e.g., the Lorentz, Poincaré, conformal groups, and differential equations

$$\mathcal{I}\, f \quad = \quad j$$

which are G-invariant. These play a very important role in the description of physical symmetries - recall, e.g., the early examples of Dirac, Maxwell, d'Allembert, equations and nowadays the latest applications of (super-)differential operators in conformal field theory, supergravity, string theory, (for a recent review, cf. e.g., [1]). Naturally, it is important to construct systematically such invariant equations and operators.

In a recent paper [2] we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

In the present paper we focus on the groups $Sp(n,I\!R)$, which are very interesting for several reasons. First of all, they belong to the class of Hermitian symmetric spaces, i.e., the pair (G,K) is a Hermitian symmetric pair (K is the maximal compact subgroup of the noncompact semisimple group G). Further, $Sp(n,I\!R)$ belong to a narrower class of groups/algebras, which we call 'conformal Lie groups or algebras' since they have very similar properties to the canonical conformal algebras $so(n,2)$ of n-dimensional Minkowski space-time. This class was identified from our point of view in [3]. Besides $so(n,2)$ it includes the algebras $su(n,n)$, $sp(n,I\!R)$, $so^*(4n)$, $E_{7(-25)}$, (omitting to mention coincidences between the low-dimensional cases, cf. [3]). The corresponding groups are also called Hermitian symmetric spaces of tube type [4]. The same class was identified from different considerations in [5], where these groups/algebras were called 'conformal groups of simple Jordan algebras'. It was identified from still different considerations also in [6], where the objects of the class were called simple space-time symmetries generalizing conformal symmetry.

In our further plans it shall be very useful that (as in [2]) we follow a procedure in representation theory in which intertwining differential operators appear canonically [7] and which procedure has been generalized to the supersymmetry setting and to quantum groups.

The present paper is organized a follows. In section 2 we give the pre-

liminaries, actually recalling and adapting facts from [2]. In Section 3 we specialize to the $sp(n,\mathbb{R})$ case. In Section 4 we present some results on the multiplet classification of the representations and intertwining differential operators between them.

2 Preliminaries

Let G be a semisimple non-compact Lie group, and K a maximal compact subgroup of G. Then we have an Iwasawa decomposition $G = KA_0N_0$, where A_0 is abelian simply connected vector subgroup of G, N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Further, let M_0 be the centralizer of A_0 in K. Then the subgroup $P_0 = M_0A_0N_0$ is a minimal parabolic subgroup of G. A parabolic subgroup $P = M'A'N'$ is any subgroup of G (including G itself) which contains a minimal parabolic subgroup.

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of G [8]. For the classification of all irreducible representations it is enough to use only the so-called *cuspidal* parabolic subgroups $P = M'A'N'$, singled out by the condition that $\mathrm{rank}\, M' = \mathrm{rank}\, M' \cap K$ [9, 10], so that M' has discrete series representations [11]. However, often induction from non-cuspidal parabolics is also convenient, cf. [12, 2, 13, 14].

Let ν be a (non-unitary) character of A', $\nu \in \mathcal{A}'^*$, let μ fix an irreducible representation D^μ of M' on a vector space V_μ.

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [15]. (These are called *generalized principal series representations* (or *limits thereof*) in [16].) Their spaces of functions are:

$$\mathcal{C}_\chi \ = \ \{\mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) \ = \ e^{-\nu(H)} \cdot D^\mu(m^{-1})\, \mathcal{F}(g)\} \qquad (1)$$

where $a = \exp(H) \in A'$, $H \in \mathcal{A}'$, $m \in M'$, $n \in N'$. The representation action is the $left$ regular action:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') \ = \ \mathcal{F}(g^{-1}g') \ , \quad g, g' \in G \ . \qquad (2)$$

For our purposes we need to restrict to *maximal* parabolic subgroups P, (so that $\mathrm{rank}\, A' = 1$), that may not be cuspidal. For the representations that we consider the character ν is parameterized by a real number d, called the conformal weight or energy.

Further, let μ fix a discrete series representation D^μ of M' on the Hilbert space V_μ, or the so-called limit of a discrete series representation (cf. [16]). Actually, instead of the discrete series we can use the finite-dimensional (non-unitary) representation of M' with the same Casimirs.

An important ingredient in our considerations are the *highest/lowest weight representations* of $\mathcal{G}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathcal{G}^{\mathbb{C}}$, where $\Lambda \in (\mathcal{H}^{\mathbb{C}})^*$, $\mathcal{H}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [7]. In this setting we can consider also unitarity, which here means positivity w.r.t. the Shapovalov form in which the conjugation is the one singling out $\mathcal{G}$ from $\mathcal{G}^{\mathbb{C}}$.

Actually, since our ERs may be induced from finite-dimensional representations of $\mathcal{M}'$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector v_0 is taken by the (finite-dimensional) space $V_\mu v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [17, 7]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^{\mathbb{C}}$, $m \in \mathbb{N}$, such that the BGG [18] Verma module reducibility condition (for highest weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) \;=\; m\ , \quad \beta^\vee \equiv 2\beta/(\beta,\beta)\ . \tag{3}$$

When (3) holds then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and β non-compact) is embedded in the Verma module V^Λ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector v_s determined by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\, v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^{\mathbb{C}}$ generated by the negative root generators [19].

More explicitly, [7], $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, v_0$ (or $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, V_\mu\, v_0$ for GVMs).[2] Then there exists [7] an intertwining differential operator

$$\mathcal{D}_{m,\beta} \;:\; \mathcal{C}_{\chi(\Lambda)} \;\longrightarrow\; \mathcal{C}_{\chi(\Lambda - m\beta)} \tag{4}$$

given explicitly by:

$$\mathcal{D}_{m,\beta} \;=\; \mathcal{P}_{m,\beta}(\widehat{\mathcal{G}^-}) \tag{5}$$

where $\widehat{\mathcal{G}^-}$ denotes the *right* action on the functions $\mathcal{F}$, cf. (1).

3 The Non-Compact Lie Algebras $sp(n, I\!\!R)$

Let $n \geq 2$. Let $\mathcal{G} = sp(n, I\!\!R)$, the split real form of $sp(n, \mathbb{C}) = \mathcal{G}^{\mathbb{C}}$. The maximal compact subgroup of $\mathcal{G}$ is $\mathcal{K} \cong u(1) \oplus su(n)$, $\dim_{I\!\!R} \mathcal{P} = n(n+1)$, $\dim_{I\!\!R} \mathcal{N} = n^2$. This algebra has discrete series representations and highest/lowest weight representations.

The split rank is equal to n, while $\mathcal{M} = 0$.

The Satake diagram [21] of $sp(n, I\!\!R)$ is the same as the Dynkin diagram of $sp(n, \mathbb{C})$:

$$\underset{\alpha_1}{\circ} \;---\; \underset{\alpha_2}{\circ} \;---\; \ldots \;---\; \underset{\alpha_{n-1}}{\circ} \;\Longleftarrow\; \underset{\alpha_n}{\circ}$$

Also the root systems coincide.

We choose a *maximal* parabolic $\mathcal{P} = \mathcal{M}'\mathcal{A}'\mathcal{N}'$ such that $\mathcal{A}' \cong so(1,1)$, while the factor $\mathcal{M}'$ has the same finite-dimensional (nonunitary) representations as the finite-dimensional (unitary) representations of the semi-simple subalgebra of $\mathcal{K}$, i.e., $\mathcal{M}' = sl(n, I\!\!R)$, cf. [2]. Thus, these induced representations are representations of finite $\mathcal{K}$-type [22]. Relatedly, the number of ERs in the corresponding multiplets is equal to $|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \;=\; 2^n$, cf. [23], where $\mathcal{H}$ is a Cartan subalgebra of both $\mathcal{G}$ and $\mathcal{K}$. Note also that $\mathcal{K}^{\mathbb{C}} \cong u(1)^{\mathbb{C}} \oplus sl(n, \mathbb{C}) \cong \mathcal{M}'^{\mathbb{C}} \oplus \mathcal{A}'^{\mathbb{C}}$. Finally, note that $\dim_{I\!\!R} \mathcal{N}' = n(n+1)/2$.

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi \;=\; \{\, n_1\,, \ldots, \, n_{n-1}\,;\, c\,\}\;, \qquad n_j \in I\!\!N\;, \quad c = d - (n+1)/2 \tag{6}$$

where the last entry of χ labels the characters of $\mathcal{A}'$, and the first $n-1$ entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M}'$, (or of the finite-dimensional unitary irreps of $su(n)$).

[2] For explicit expressions for singular vectors we refer to [20].

The reason to use the parameter c instead of d is that the parametrization of the ERs in the multiplets is given in a simpler way, as we shall see.

Below we shall use the following conjugation on the finite-dimensional entries of the signature:

$$(n_1, \ldots, n_{n-1})^* \doteq (n_{n-1}, \ldots, n_1) \tag{7}$$

The ERs in the multiplet are related also by intertwining integral operators. The integral operators were introduced by Knapp and Stein [24]. In fact, these operators are defined for any ER, not only for the reducible ones, the general action being:

$$\begin{aligned} G_{KS} &: \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'} , \\ \chi &= \{ n_1, \ldots, n_{n-1} ; c \} , \\ \chi' &= \{ (n_1, \ldots, n_{n-1})^* ; -c \} \end{aligned} \tag{8}$$

The above action on the signatures is also called restricted Weyl reflection, since it represents the nontrivial element of the 2-element restricted Weyl group which arises canonically with every maximal parabolic subalgebra.[3]

Further, we need more explicitly the root system of the algebra $sp(n, F)$.

In terms of the orthonormal basis ϵ_i , $i = 1, \ldots, n$, the positive roots are given by

$$\Delta^+ = \{\epsilon_i \pm \epsilon_j,\ 1 \leq i < j \leq n;\ 2\epsilon_i, 1 \leq i \leq n\}, \tag{9}$$

while the simple roots are:

$$\pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1},\ 1 \leq i \leq n-1;\ \alpha_n = 2\epsilon_n\} \tag{10}$$

With our choice of normalization of the long roots $2\epsilon_k$ have length 4, while the short roots $\epsilon_i \pm \epsilon_j$ have length 2.

From these the compact roots are those that form (by restriction) the root system of the semisimple part of $\mathcal{K}^{\mathbb{C}}$, the rest are noncompact, i.e.,

$$\begin{aligned} \text{compact} : &\quad \alpha_{ij} \equiv \epsilon_i - \epsilon_j, \ , \quad 1 \leq i < j \leq n \ , \\ \text{noncompact} : &\quad \beta_{ij} \equiv \epsilon_i + \epsilon_j, \ , \quad 1 \leq i \leq j \leq n \end{aligned} \tag{11}$$

Thus, the only non-compact simple root is $\alpha_n = \beta_{nn}$.

[3]Generically, the Knapp-Stein operators can be normalized so that indeed $G_{KS} \circ G_{KS} = \mathrm{Id}_{\mathcal{C}_\chi}$. However, this usually fails exactly for the reducible ERs that form the multiplets, cf., e.g., [15].

We adopt the following ordering of the roots:

$$\begin{array}{ccccccccc}
\beta_{11} & & & & & & & & \\
\vee & & & & & & & & \\
\beta_{12} & > & \beta_{22} & & & & & & \\
\vee & & \vee & & & & & & \\
\dots & \dots & \dots & \dots & \dots & & & & \\
\vee & & \vee & & \dots & & \vee & & \\
\beta_{1n} & > & \beta_{2n} & > & \dots & > & \beta_{n-1,n} & > & \beta_{nn} = \alpha_n \\
\vee & & \vee & & \dots & & \vee & & \\
\alpha_{1n} & > & \alpha_{2n} & > & \dots & > & \alpha_{n-1,n} = \alpha_{n-1} & & \\
\vee & & \vee & & \dots & & & & \\
\dots & \dots & \dots & \dots & \dots & & & & \\
\vee & & \vee & & & & & & \\
\alpha_{13} & > & \alpha_{23} = \alpha_2 & & & & & & \\
\vee & & & & & & & & \\
\alpha_{12} = \alpha_1 & & & & & & & &
\end{array} \tag{12}$$

This ordering is lexicographical adopting the ordering of the ϵ :

$$\epsilon_1 > \cdots > \epsilon_n \tag{13}$$

Further, we shall use the so-called Dynkin labels:

$$m_i \;\equiv\; (\Lambda + \rho, \alpha_i^\vee) \;, \quad i = 1, \dots, n, \tag{14}$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$.

We shall use also the so-called Harish-Chandra parameters:

$$m_\beta \equiv (\Lambda + \rho, \beta) \;, \tag{15}$$

where β is any positive root of $\mathcal{G}^{\mathbb{C}}$. These parameters are redundant, since they are expressed in terms of the Dynkin labels, however, some statements are best formulated in their terms. In particular, in the case of the noncompact roots we have:

$$\begin{aligned}
m_{\beta_{ij}} &= \Big(\sum_{s=i}^{n} + \sum_{s=j}^{n} \Big) m_s \;, \qquad i < j \;, \\
m_{\beta_{ii}} &= \sum_{s=i}^{n} m_s
\end{aligned} \tag{16}$$

Now we can give the correspondence between the signatures χ and the highest weight Λ. The explicit connection is:

$$n_i = m_i \ , \quad c \ = \ -\tfrac{1}{2}(m_{\tilde{\alpha}} + m_n) \ = \ -\tfrac{1}{2}(m_1 + \cdots + m_{n-1} + 2m_n) \qquad (17)$$

where $\tilde{\alpha} = \beta_{11}$ is the highest root.

There are several types of multiplets: the main type, (which contains maximal number of ERs/GVMs, the finite-dimensional and the discrete series representations), and some reduced types of multiplets.

In the next Section we give the main type of multiplets and the main reduced types for $sp(n, I\!\!R)$ for $n \leq 6$.

4 Multiplets

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $sp(n, I\!\!R)$, i.e., they will be labelled by the n positive Dynkin labels $m_i \in I\!\!N$. As we mentioned, each such multiplet contains 2^n ERs/GVMs. It is difficult to give explicitly the multiplets for general n. Thus, we shall give explicitly the case $n = 6$ which can still be represented and comprehended, and then show how to obtain the cases $n < 6$.

4.1 sp(6,$I\!\!R$)

4.1.1 Main multiplets

The main multiplets R^6 contain $64 (= 2^6)$ ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^{\pm} &= \{ (m_1, m_2, m_3, m_4, m_5)^{\pm} \, ; \, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} + m_6) \} \qquad (18)\\
\chi_a^{\pm} &= \{ (m_1, m_2, m_3, m_4, m_5 + 2m_6)^{\pm} \, ; \, \pm\tfrac{1}{2}m_{15} \} \\
\chi_b^{\pm} &= \{ (m_1, m_2, m_3, m_{45}, m_5 + 2m_6)^{\pm} \, ; \, \pm\tfrac{1}{2}m_{14} \} \\
\chi_c^{\pm} &= \{ (m_1, m_2, m_{34}, m_5, m_{45} + 2m_6)^{\pm} \, ; \, \pm\tfrac{1}{2}m_{13} \} \\
\chi_{c'}^{\pm} &= \{ (m_1, m_2, m_3, m_{45} + 2m_6, m_5)^{\pm} \, ; \, \pm\tfrac{1}{2}m_{14} \} \\
\chi_d^{\pm} &= \{ (m_1, m_{23}, m_4, m_5, m_{35} + 2m_6)^{\pm} \, ; \, \pm\tfrac{1}{2}m_{12} \} \\
\chi_{d'}^{\pm} &= \{ (m_1, m_2, m_{34}, m_5 + 2m_6, m_{45})^{\pm} \, ; \, \pm\tfrac{1}{2}m_{13} \} \\
\chi_e^{\pm} &= \{ (m_{12}, m_3, m_4, m_5, m_{25} + 2m_6)^{\pm} \, ; \, \pm\tfrac{1}{2}m_1 \} \\
\chi_{e'}^{\pm} &= \{ (m_1, m_{23}, m_4, m_5 + 2m_6, m_{35})^{\pm} \, ; \, \pm\tfrac{1}{2}m_{12} \} \\
\chi_{e''}^{\pm} &= \{ (m_1, m_2, m_{35}, m_5 + 2m_6, m_4)^{\pm} \, ; \, \pm\tfrac{1}{2}m_{13} \}
\end{aligned}$$

$$
\begin{array}{rcl}
\chi^\pm_f &=& \{ (m_2, m_3, m_4, m_5, m_{15} + 2m_6)^\pm ; \mp\frac{1}{2}m_1 \} \\
\chi^\pm_{f'} &=& \{ (m_{12}, m_3, m_4, m_5 + 2m_6, m_{25})^\pm ; \pm\frac{1}{2}m_1 \} \\
\chi^\pm_{f''} &=& \{ (m_1, m_{23}, m_{45}, m_5 + 2m_6, m_{34})^\pm ; \pm\frac{1}{2}m_{12} \} \\
\chi^\pm_{f'''} &=& \{ (m_1, m_2, m_{35} + 2m_6, m_5, m_4)^\pm ; \pm\frac{1}{2}m_{13} \} \\
\chi^\pm_g &=& \{ (m_2, m_3, m_4, m_5 + 2m_6, m_{15})^\pm ; \mp\frac{1}{2}m_1 \} \\
\chi^\pm_{g'} &=& \{ (m_{12}, m_3, m_{45}, m_5 + 2m_6, m_{24})^\pm ; \pm\frac{1}{2}m_1 \} \\
\chi^\pm_{g''} &=& \{ (m_1, m_{23}, m_{45} + 2m_6, m_5, m_{34})^\pm ; \pm\frac{1}{2}m_{12} \} \\
\chi^\pm_h &=& \{ (m_2, m_3, m_{45}, m_5 + 2m_6, m_{14})^\pm ; \mp\frac{1}{2}m_1 \} \\
\chi^\pm_{h'} &=& \{ (m_{12}, m_3, m_{45} + 2m_6, m_5, m_{24})^\pm ; \pm\frac{1}{2}m_1 \} \\
\chi^\pm_{h''} &=& \{ (m_2, m_3, m_{45} + 2m_6, m_5, m_{14})^\pm ; \mp\frac{1}{2}m_1 \} \\
\chi^\pm_j &=& \{ (m_2, m_{34}, m_5, m_{45} + 2m_6, m_{13})^\pm ; \mp\frac{1}{2}m_1 \} \\
\chi^\pm_{j'} &=& \{ (m_{12}, m_{34}, m_5, m_{45} + 2m_6, m_{23})^\pm ; \pm\frac{1}{2}m_1 \} \\
\chi^\pm_{j''} &=& \{ (m_1, m_{24}, m_5, m_{45} + 2m_6, m_3)^\pm ; \pm\frac{1}{2}m_{12} \} \\
\chi^\pm_k &=& \{ (m_2, m_{34}, m_5 + 2m_6, m_{45}, m_{13})^\pm ; \mp\frac{1}{2}m_1 \} \\
\chi^\pm_{k'} &=& \{ (m_{12}, m_{34}, m_5 + 2m_6, m_{45}, m_{23})^\pm ; \pm\frac{1}{2}m_1 \} \\
\chi^\pm_{k''} &=& \{ (m_1, m_{24}, m_5 + 2m_6, m_{45}, m_3)^\pm ; \pm\frac{1}{2}m_{12} \} \\
\chi^\pm_\ell &=& \{ (m_2, m_{35}, m_5 + 2m_6, m_4, m_{13})^\pm ; \mp\frac{1}{2}m_1 \} \\
\chi^\pm_{\ell'} &=& \{ (m_{12}, m_{35}, m_5 + 2m_6, m_4, m_{23})^\pm ; \pm\frac{1}{2}m_1 \} \\
\chi^\pm_{\ell''} &=& \{ (m_1, m_{25}, m_5 + 2m_6, m_4, m_3)^\pm ; \pm\frac{1}{2}m_{12} \} \\
\chi^\pm_m &=& \{ (m_2, m_{35} + 2m_6, m_5, m_4, m_{13})^\pm ; \mp\frac{1}{2}m_1 \} \\
\chi^\pm_{m'} &=& \{ (m_{12}, m_{35} + 2m_6, m_5, m_4, m_{23})^\pm ; \pm\frac{1}{2}m_1 \} \\
\chi^\pm_{m''} &=& \{ (m_1, m_{25} + 2m_6, m_5, m_4, m_3)^\pm ; \pm\frac{1}{2}m_{12} \}
\end{array}
$$

where the notation $(...)^\pm$ employs the conjugation (7) :

$$(n_1, ..., n_5)^- = (n_1, ..., n_5) , \qquad (n_1, ..., n_5)^+ = (n_1, ..., n_5)^* = (n_5, ..., n_1) \tag{19}$$

Obviously, the pairs in (18) are related by Knapp-Stein integral operators, i.e.,

$$G_{KS} \ : \ \mathcal{C}_{\chi^\mp} \longleftrightarrow \mathcal{C}_{\chi^\pm} \tag{20}$$

Matters are arranged so that in every multiplet only the ER with signature χ_0^- contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of $sp(6)$ with signature $\{m_1 , \dots , m_6\}$. The subspace $\mathcal{E}$ is annihilated by the operator G^+, and is the image of the operator G^-. The subspace $\mathcal{E}$ is annihilated also by the intertwining differential operator acting from χ^- to χ'^- (more about this operator below). When all $m_i = 1$ then

dim $\mathcal{E} = 1$, and in that case $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $\mathcal{G}$. Furthermore in that case the conformal weight is zero: $d = \frac{7}{2} + c = \frac{7}{2} - \frac{1}{2}(m_1 + \cdots + m_5 + 2m_6)_{|m_i=1} = 0$.

Analogously, in every multiplet only the ER with signature χ_0^+ contains holomorphic discrete series representation. This is guaranteed by the criterion [11] that for such an ER all Harish-Chandra parameters for non-compact roots must be negative, i.e., in our situation, $m_\alpha < 0$, for α from the second row of (11). [That this holds for our χ^+ can be easily checked using the signatures (18).]

In fact, the Harish-Chandra parameters are reflected in the division of the ERs into χ^- and χ^+ : for the χ^- less than half of the 21 non-compact Harish-Chandra parameters are negative, (none for χ_0^-), while for the χ^+ more than half of the 21 non-compact Harish-Chandra parameters are negative, (all for χ_0^+),

Note that the ER χ_0^+ contains also the conjugate anti-holomorphic discrete series. The direct sum of the holomorphic and the antiholomorphic representations are realized in an invariant subspace $\mathcal{D}$ of the ER χ_0^+. That subspace is annihilated by the operator G^-, and is the image of the operator G^+.

Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

The conformal weight of the ER χ_0^+ has the restriction $d = \frac{7}{2} + c = \frac{7}{2} + \frac{1}{2}(m_1 + \cdots + m_5 + 2m_6) \geq 7$.

The multiplets are given explicitly in Fig. 1, where we use the notation: $\Lambda^\pm = \Lambda(\chi^\pm)$. Each intertwining differential operator is represented by an arrow accompanied by a symbol $i_{j...k}$ encoding the root $\beta_{j...k}$ and the number $m_{\beta_{j...k}}$ which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators which are non-composite are displayed, and that the data β, m_β, which is involved in the embedding $V^\Lambda \longleftrightarrow V^{\Lambda - m_\beta,\beta}$ turns out to involve only the m_i corresponding to simple roots, i.e., for each β, m_β there exists $i = i(\beta, m_\beta, \Lambda) \in \{1, \ldots, 2n-1\}$, such that $m_\beta = m_i$. Hence the data $\beta_{j...k}$, $m_{\beta_{j...k}}$ is represented by $i_{j...k}$ on the arrows.

The pairs $\Lambda^\pm$ are symmetric w.r.t. to the bullet in the middle of the figure - this represents the Weyl symmetry realized by the Knapp-Stein operators.

4.1.2 Reduced multiplets R_1^6

The reduced multiplets of type R_1^6 contain 48 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{aligned}
\chi_0^\pm &= \{ (0, m_2, m_3, m_4, m_5)^\pm ; \pm\tfrac{1}{2}(m_{25} + 2m_6) \} \\
\chi_a^\pm &= \{ (0, m_2, m_3, m_4, m_5 + 2m_6)^\pm ; \pm\tfrac{1}{2}m_{25} \} \\
\chi_b^\pm &= \{ (0, m_2, m_3, m_{45}, m_5 + 2m_6)^\pm ; \pm\tfrac{1}{2}m_{24} \} \\
\chi_c^\pm &= \{ (0, m_2, m_{34}, m_5, m_{45} + 2m_6)^\pm ; \pm\tfrac{1}{2}m_{23} \} \\
\chi_{c'}^\pm &= \{ (0, m_2, m_3, m_{45} + 2m_6, m_5)^\pm ; \pm\tfrac{1}{2}m_{24} \} \\
\chi_d^\pm &= \{ (0, m_{23}, m_4, m_5, m_{35} + 2m_6)^\pm ; \pm\tfrac{1}{2}m_2 \} \\
\chi_{d'}^\pm &= \{ (0, m_2, m_{34}, m_5 + 2m_6, m_{45})^\pm ; \pm\tfrac{1}{2}m_{23} \} \\
\chi_e^\pm &= \{ (m_2, m_3, m_4, m_5, m_{25} + 2m_6)^\pm ; 0 \} \\
\chi_{e'}^\pm &= \{ (0, m_{23}, m_4, m_5 + 2m_6, m_{35})^\pm ; \pm\tfrac{1}{2}m_2 \} \\
\chi_{e''}^\pm &= \{ (0, m_2, m_{35}, m_5 + 2m_6, m_4)^\pm ; \pm\tfrac{1}{2}m_{23} \} \\
\chi_{f'}^\pm &= \{ (m_2, m_3, m_4, m_5 + 2m_6, m_{25})^\pm ; 0 \} \\
\chi_{f''}^\pm &= \{ (0, m_{23}, m_{45}, m_5 + 2m_6, m_{34})^\pm ; \pm\tfrac{1}{2}m_2 \} \\
\chi_{f'''}^\pm &= \{ (0, m_2, m_{35} + 2m_6, m_5, m_4)^\pm ; \pm\tfrac{1}{2}m_{23} \} \\
\chi_{g'}^\pm &= \{ (m_2, m_3, m_{45}, m_5 + 2m_6, m_{24})^\pm ; 0 \} \\
\chi_{g''}^\pm &= \{ (0, m_{23}, m_{45} + 2m_6, m_5, m_{34})^\pm ; \pm\tfrac{1}{2}m_2 \} \\
\chi_{h'}^\pm &= \{ (m_2, m_3, m_{45} + 2m_6, m_5, m_{24})^\pm ; 0 \} \\
\chi_{j'}^\pm &= \{ (m_2, m_{34}, m_5, m_{45} + 2m_6, m_{23})^\pm ; 0 \} \\
\chi_{j''}^\pm &= \{ (0, m_{24}, m_5, m_{45} + 2m_6, m_3)^\pm ; \pm\tfrac{1}{2}m_2 \} \\
\chi_{k'}^\pm &= \{ (m_2, m_{34}, m_5 + 2m_6, m_{45}, m_{23})^\pm ; 0 \} \\
\chi_{k''}^\pm &= \{ (0, m_{24}, m_5 + 2m_6, m_{45}, m_3)^\pm ; \pm\tfrac{1}{2}m_2 \} \\
\chi_{\ell'}^\pm &= \{ (m_2, m_{35}, m_5 + 2m_6, m_4, m_{23})^\pm ; 0 \} \\
\chi_{\ell''}^\pm &= \{ (0, m_{25}, m_5 + 2m_6, m_4, m_3)^\pm ; \pm\tfrac{1}{2}m_2 \} \\
\chi_{m'}^\pm &= \{ (m_2, m_{35} + 2m_6, m_5, m_4, m_{23})^\pm ; 0 \} \\
\chi_{m''}^\pm &= \{ (0, m_{25} + 2m_6, m_5, m_4, m_3)^\pm ; \pm\tfrac{1}{2}m_2 \}
\end{aligned}
\tag{21}
$$

The multiplets are given explicitly in Fig. 1a.

4.1.3 Reduced multiplets R_2^6

The reduced multiplets of type R_2^6 contain 48 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{aligned}
\chi_0^\pm &= \{ (m_1, 0, m_3, m_4, m_5)^\pm ; \pm\tfrac{1}{2}(m_{1,35} + 2m_6) \} \\
\chi_a^\pm &= \{ (m_1, 0, m_3, m_4, m_5 + 2m_6)^\pm ; \pm\tfrac{1}{2}m_{1,35} \}
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\chi_b^\pm &= \{ (m_1, 0, m_3, m_{45}, m_5 + 2m_6)^\pm ; \pm\tfrac{1}{2} m_{1,34} \} \\
\chi_c^\pm &= \{ (m_1, 0, m_{34}, m_5, m_{45} + 2m_6)^\pm ; \pm\tfrac{1}{2} m_{1,3} \} \\
\chi_{c'}^\pm &= \{ (m_1, 0, m_3, m_{45} + 2m_6, m_5)^\pm ; \pm\tfrac{1}{2} m_{1,34} \} \\
\chi_d^\pm &= \{ (m_1, m_3, m_4, m_5, m_{35} + 2m_6)^\pm ; \pm\tfrac{1}{2} m_1 \} \\
\chi_{d'}^\pm &= \{ (m_1, 0, m_{34}, m_5 + 2m_6, m_{45})^\pm ; \pm\tfrac{1}{2} m_{1,3} \} \\
\chi_{e'}^\pm &= \{ (m_1, m_3, m_4, m_5 + 2m_6, m_{35})^\pm ; \pm\tfrac{1}{2} m_1 \} \\
\chi_{e''}^\pm &= \{ (m_1, 0, m_{35}, m_5 + 2m_6, m_4)^\pm ; \pm\tfrac{1}{2} m_{1,3} \} \\
\chi_f^\pm &= \{ (0, m_3, m_4, m_5, m_{1,35} + 2m_6)^\pm ; \mp\tfrac{1}{2} m_1 \} \\
\chi_{f''}^\pm &= \{ (m_1, m_3, m_{45}, m_5 + 2m_6, m_{34})^\pm ; \pm\tfrac{1}{2} m_1 \} \\
\chi_{f'''}^\pm &= \{ (m_1, 0, m_{35} + 2m_6, m_5, m_4)^\pm ; \pm\tfrac{1}{2} m_{1,3} \} \\
\chi_g^\pm &= \{ (0, m_3, m_4, m_5 + 2m_6, m_{1,35})^\pm ; \mp\tfrac{1}{2} m_1 \} \\
\chi_{g''}^\pm &= \{ (m_1, m_3, m_{45} + 2m_6, m_5, m_{34})^\pm ; \pm\tfrac{1}{2} m_1 \} \\
\chi_h^\pm &= \{ (0, m_3, m_{45}, m_5 + 2m_6, m_{1,34})^\pm ; \mp\tfrac{1}{2} m_1 \} \\
\chi_{h''}^\pm &= \{ (0, m_3, m_{45} + 2m_6, m_5, m_{1,34})^\pm ; \mp\tfrac{1}{2} m_1 \} \\
\chi_j^\pm &= \{ (0, m_{34}, m_5, m_{45} + 2m_6, m_{1,3})^\pm ; \mp\tfrac{1}{2} m_1 \} \\
\chi_{j''}^\pm &= \{ (m_1, m_{34}, m_5, m_{45} + 2m_6, m_3)^\pm ; \pm\tfrac{1}{2} m_1 \} \\
\chi_k^\pm &= \{ (0, m_{34}, m_5 + 2m_6, m_{45}, m_{1,3})^\pm ; \mp\tfrac{1}{2} m_1 \} \\
\chi_{k''}^\pm &= \{ (m_1, m_{34}, m_5 + 2m_6, m_{45}, m_3)^\pm ; \pm\tfrac{1}{2} m_1 \} \\
\chi_\ell^\pm &= \{ (0, m_{35}, m_5 + 2m_6, m_4, m_{1,3})^\pm ; \mp\tfrac{1}{2} m_1 \} \\
\chi_{\ell''}^\pm &= \{ (m_1, m_{35}, m_5 + 2m_6, m_4, m_3)^\pm ; \pm\tfrac{1}{2} m_1 \} \\
\chi_m^\pm &= \{ (0, m_{35} + 2m_6, m_5, m_4, m_{1,3})^\pm ; \mp\tfrac{1}{2} m_1 \} \\
\chi_{m''}^\pm &= \{ (m_1, m_{35} + 2m_6, m_5, m_4, m_3)^\pm ; \pm\tfrac{1}{2} m_1 \}
\end{aligned}
$$

The multiplets are given explicitly in Fig. 1b.

4.1.4 Reduced multiplets R_3^6

The reduced multiplets of type R_3^6 contain 48 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{aligned}
\chi_0^\pm &= \{ (m_1, m_2, 0, m_4, m_5)^\pm ; \pm\tfrac{1}{2}(m_{12,45} + 2m_6) \} \qquad (23) \\
\chi_a^\pm &= \{ (m_1, m_2, 0, m_4, m_5 + 2m_6)^\pm ; \pm\tfrac{1}{2} m_{12,45} \} \\
\chi_b^\pm &= \{ (m_1, m_2, 0, m_{45}, m_5 + 2m_6)^\pm ; \pm\tfrac{1}{2} m_{12,4} \} \\
\chi_{c'}^\pm &= \{ (m_1, m_2, 0, m_{45} + 2m_6, m_5)^\pm ; \pm\tfrac{1}{2} m_{12,4} \} \\
\chi_d^\pm &= \{ (m_1, m_2, m_4, m_5, m_{45} + 2m_6)^\pm ; \pm\tfrac{1}{2} m_{12} \} \\
\chi_e^\pm &= \{ (m_{12}, 0, m_4, m_5, m_{2,45} + 2m_6)^\pm ; \pm\tfrac{1}{2} m_1 \} \\
\chi_{e'}^\pm &= \{ (m_1, m_2, m_4, m_5 + 2m_6, m_{45})^\pm ; \pm\tfrac{1}{2} m_{12} \} \\
\chi_f^\pm &= \{ (m_2, 0, m_4, m_5, m_{12,45} + 2m_6)^\pm ; \mp\tfrac{1}{2} m_1 \}
\end{aligned}
$$

$$
\begin{array}{rcl}
\chi^{\pm}_{f'} &=& \{\, (m_{12}, 0, m_4, m_5 + 2m_6, m_{2,45})^{\pm}\, ;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{f''} &=& \{\, (m_1, m_2, m_{45}, m_5 + 2m_6, m_4)^{\pm}\, ;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi^{\pm}_{g} &=& \{\, (m_2, 0, m_4, m_5 + 2m_6, m_{12,45})^{\pm}\, ;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{g'} &=& \{\, (m_{12}, 0, m_{45}, m_5 + 2m_6, m_{2,4})^{\pm}\, ;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{g''} &=& \{\, (m_1, m_2, m_{45} + 2m_6, m_5, m_4)^{\pm}\, ;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi^{\pm}_{h} &=& \{\, (m_2, 0, m_{45}, m_5 + 2m_6, m_{12,4})^{\pm}\, ;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{h'} &=& \{\, (m_{12}, 0, m_{45} + 2m_6, m_5, m_{2,4})^{\pm}\, ;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{h''} &=& \{\, (m_2, 0, m_{45} + 2m_6, m_5, m_{12,4})^{\pm}\, ;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{j} &=& \{\, (m_2, m_4, m_5, m_{45} + 2m_6, m_{12})^{\pm}\, ;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{j'} &=& \{\, (m_{12}, m_4, m_5, m_{45} + 2m_6, m_2)^{\pm}\, ;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{j''} &=& \{\, (m_1, m_{2,4}, m_5, m_{45} + 2m_6, 0)^{\pm}\, ;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi^{\pm}_{k} &=& \{\, (m_2, m_4, m_5 + 2m_6, m_{45}, m_{12})^{\pm}\, ;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{k'} &=& \{\, (m_{12}, m_4, m_5 + 2m_6, m_{45}, m_2)^{\pm}\, ;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{k''} &=& \{\, (m_1, m_{2,4}, m_5 + 2m_6, m_{45}, 0)^{\pm}\, ;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi^{\pm}_{\ell} &=& \{\, (m_2, m_{45}, m_5 + 2m_6, m_4, m_{12})^{\pm}\, ;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{\ell'} &=& \{\, (m_{12}, m_{45}, m_5 + 2m_6, m_4, m_2)^{\pm}\, ;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{\ell''} &=& \{\, (m_1, m_{2,45}, m_5 + 2m_6, m_4, 0)^{\pm}\, ;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi^{\pm}_{m} &=& \{\, (m_2, m_{45} + 2m_6, m_5, m_4, m_{12})^{\pm}\, ;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{m'} &=& \{\, (m_{12}, m_{45} + 2m_6, m_5, m_4, m_2)^{\pm}\, ;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{m''} &=& \{\, (m_1, m_{2,45} + 2m_6, m_5, m_4, 0)^{\pm}\, ;\, \pm\tfrac{1}{2} m_{12} \,\}
\end{array}
$$

The multiplets are given explicitly in Fig. 1c.

4.1.5 Reduced multiplets R^6_4

The reduced multiplets of type R^6_4 contain 48 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{array}{rcl}
\chi^{\pm}_{0} &=& \{\, (m_1, m_2, m_3, 0, m_5)^{\pm}\, ;\, \pm\tfrac{1}{2}(m_{13,5} + 2m_6) \,\} \\
\chi^{\pm}_{a} &=& \{\, (m_1, m_2, m_3, 0, m_5 + 2m_6)^{\pm}\, ;\, \pm\tfrac{1}{2} m_{13,5} \,\} \\
\chi^{\pm}_{c} &=& \{\, (m_1, m_2, m_3, m_5, m_5 + 2m_6)^{\pm}\, ;\, \pm\tfrac{1}{2} m_{13} \,\} \\
\chi^{\pm}_{d} &=& \{\, (m_1, m_{23}, 0, m_5, m_{3,5} + 2m_6)^{\pm}\, ;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi^{\pm}_{d'} &=& \{\, (m_1, m_2, m_3, m_5 + 2m_6, m_5)^{\pm}\, ;\, \pm\tfrac{1}{2} m_{13} \,\} \\
\chi^{\pm}_{e} &=& \{\, (m_{12}, m_3, 0, m_5, m_{23,5} + 2m_6)^{\pm}\, ;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{e'} &=& \{\, (m_1, m_{23}, 0, m_5 + 2m_6, m_{3,5})^{\pm}\, ;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi^{\pm}_{e''} &=& \{\, (m_1, m_2, m_{3,5}, m_5 + 2m_6, 0)^{\pm}\, ;\, \pm\tfrac{1}{2} m_{13} \,\} \\
\chi^{\pm}_{f} &=& \{\, (m_2, m_3, 0, m_5, m_{13,5} + 2m_6)^{\pm}\, ;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^{\pm}_{f'} &=& \{\, (m_{12}, m_3, 0, m_5 + 2m_6, m_{23,5})^{\pm}\, ;\, \pm\tfrac{1}{2} m_1 \,\}
\end{array}
\qquad (24)
$$

$$\begin{array}{rcl}
\chi^\pm_{f'''} &=& \{\,(m_1,m_2,m_{3,5}+2m_6,m_5,0)^\pm\,;\,\pm\frac12 m_{13}\,\}\\
\chi^\pm_{g} &=& \{\,(m_2,m_3,0,m_5+2m_6,m_{13,5})^\pm\,;\,\mp\frac12 m_{1}\,\}\\
\chi^\pm_{j} &=& \{\,(m_2,m_3,m_5,m_5+2m_6,m_{13})^\pm\,;\,\mp\frac12 m_{1}\,\}\\
\chi^\pm_{j'} &=& \{\,(m_{12},m_3,m_5,m_5+2m_6,m_{23})^\pm\,;\,\pm\frac12 m_{1}\,\}\\
\chi^\pm_{j''} &=& \{\,(m_1,m_{23},m_5,m_5+2m_6,m_{3})^\pm\,;\,\pm\frac12 m_{12}\,\}\\
\chi^\pm_{k} &=& \{\,(m_2,m_3,m_5+2m_6,m_5,m_{13})^\pm\,;\,\mp\frac12 m_{1}\,\}\\
\chi^\pm_{k'} &=& \{\,(m_{12},m_3,m_5+2m_6,m_5,m_{23})^\pm\,;\,\pm\frac12 m_{1}\,\}\\
\chi^\pm_{k''} &=& \{\,(m_1,m_{23},m_5+2m_6,m_5,m_{3})^\pm\,;\,\pm\frac12 m_{12}\,\}\\
\chi^\pm_{\ell} &=& \{\,(m_2,m_{3,5},m_5+2m_6,0,m_{13})^\pm\,;\,\mp\frac12 m_{1}\,\}\\
\chi^\pm_{\ell'} &=& \{\,(m_{12},m_{3,5},m_5+2m_6,0,m_{23})^\pm\,;\,\pm\frac12 m_{1}\,\}\\
\chi^\pm_{\ell''} &=& \{\,(m_1,m_{23,5},m_5+2m_6,0,m_{3})^\pm\,;\,\pm\frac12 m_{12}\,\}\\
\chi^\pm_{m} &=& \{\,(m_2,m_{3,5}+2m_6,m_5,0,m_{13})^\pm\,;\,\mp\frac12 m_{1}\,\}\\
\chi^\pm_{m'} &=& \{\,(m_{12},m_{3,5}+2m_6,m_5,0,m_{23})^\pm\,;\,\pm\frac12 m_{1}\,\}\\
\chi^\pm_{m''} &=& \{\,(m_1,m_{23,5}+2m_6,m_5,0,m_{3})^\pm\,;\,\pm\frac12 m_{12}\,\}
\end{array}$$

The multiplets are given explicitly in Fig. 1d.

4.1.6 Reduced multiplets R^6_5

The reduced multiplets of type R^6_5 contain 48 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{array}{rcl}
\chi^\pm_{0} &=& \{\,(m_1,m_2,m_3,m_4,0)^\pm\,;\,\pm\frac12 (m_{14}+2m_6)\,\}\\
\chi^\pm_{a} &=& \{\,(m_1,m_2,m_3,m_4,2m_6)^\pm\,;\,\pm\frac12 m_{14}\,\}\\
\chi^\pm_{b} &=& \{\,(m_1,m_2,m_3,m_4,2m_6)^\pm\,;\,\pm\frac12 m_{14}\,\}\\
\chi^\pm_{c} &=& \{\,(m_1,m_2,m_{34},0,m_4+2m_6)^\pm\,;\,\pm\frac12 m_{13}\,\}\\
\chi^\pm_{c'} &=& \{\,(m_1,m_2,m_3,m_4+2m_6,0)^\pm\,;\,\pm\frac12 m_{14}\,\}\\
\chi^\pm_{d} &=& \{\,(m_1,m_{23},m_4,0,m_{34}+2m_6)^\pm\,;\,\pm\frac12 m_{12}\,\}\\
\chi^\pm_{d'} &=& \{\,(m_1,m_2,m_{34},2m_6,m_4)^\pm\,;\,\pm\frac12 m_{13}\,\}\\
\chi^\pm_{e} &=& \{\,(m_{12},m_3,m_4,0,m_{24}+2m_6)^\pm\,;\,\pm\frac12 m_{1}\,\}\\
\chi^\pm_{e'} &=& \{\,(m_1,m_{23},m_4,2m_6,m_{34})^\pm\,;\,\pm\frac12 m_{12}\,\}\\
\chi^\pm_{e''} &=& \{\,(m_1,m_2,m_{34},2m_6,m_4)^\pm\,;\,\pm\frac12 m_{13}\,\}\\
\chi^\pm_{f} &=& \{\,(m_2,m_3,m_4,0,m_{14}+2m_6)^\pm\,;\,\mp\frac12 m_{1}\,\}\\
\chi^\pm_{f'} &=& \{\,(m_{12},m_3,m_4,2m_6,m_{24})^\pm\,;\,\pm\frac12 m_{1}\,\}\\
\chi^\pm_{f''} &=& \{\,(m_1,m_{23},m_4,2m_6,m_{34})^\pm\,;\,\pm\frac12 m_{12}\,\}\\
\chi^\pm_{f'''} &=& \{\,(m_1,m_2,m_{34}+2m_6,0,m_4)^\pm\,;\,\pm\frac12 m_{13}\,\}\\
\chi^\pm_{g} &=& \{\,(m_2,m_3,m_4,2m_6,m_{14})^\pm\,;\,\mp\frac12 m_{1}\,\}\\
\chi^\pm_{g'} &=& \{\,(m_{12},m_3,m_4,2m_6,m_{24})^\pm\,;\,\pm\frac12 m_{1}\,\}
\end{array} \tag{25}$$

$$
\begin{aligned}
\chi^\pm_{g''} &= \{\, (m_1, m_{23}, m_4 + 2m_6, 0, m_{34})^\pm \,;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi^\pm_{h} &= \{\, (m_2, m_3, m_4, 2m_6, m_{14})^\pm \,;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{h'} &= \{\, (m_{12}, m_3, m_4 + 2m_6, 0, m_{24})^\pm \,;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{h''} &= \{\, (m_2, m_3, m_4 + 2m_6, 0, m_{14})^\pm \,;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{j} &= \{\, (m_2, m_{34}, 0, m_4 + 2m_6, m_{13})^\pm \,;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{j'} &= \{\, (m_{12}, m_{34}, 0, m_4 + 2m_6, m_{23})^\pm \,;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{j''} &= \{\, (m_1, m_{24}, 0, m_4 + 2m_6, m_3)^\pm \,;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi^\pm_{k} &= \{\, (m_2, m_{34}, 2m_6, m_4, m_{13})^\pm \,;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{k'} &= \{\, (m_{12}, m_{34}, 2m_6, m_4, m_{23})^\pm \,;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{k''} &= \{\, (m_1, m_{24}, 2m_6, m_4, m_3)^\pm \,;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi^\pm_{m} &= \{\, (m_2, m_{34} + 2m_6, 0, m_4, m_{13})^\pm \,;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{m'} &= \{\, (m_{12}, m_{34} + 2m_6, 0, m_4, m_{23})^\pm \,;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{m''} &= \{\, (m_1, m_{24} + 2m_6, 0, m_4, m_3)^\pm \,;\, \pm\tfrac{1}{2} m_{12} \,\}
\end{aligned}
$$

The multiplets are given explicitly in Fig. 1e.

4.1.7 Reduced multiplets R^6_6

The reduced multiplets of type R^6_6 contain 32 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{aligned}
\chi^\pm_{0} &= \{\, (m_1, m_2, m_3, m_4, m_5)^\pm \,;\, \pm\tfrac{1}{2} m_{15} \,\} \qquad (26) \\
\chi^\pm_{c'} &= \{\, (m_1, m_2, m_3, m_{45}, m_5)^\pm \,;\, \pm\tfrac{1}{2} m_{14} \,\} \\
\chi^\pm_{d'} &= \{\, (m_1, m_2, m_{34}, m_5, m_{45})^\pm \,;\, \pm\tfrac{1}{2} m_{13} \,\} \\
\chi^\pm_{e'} &= \{\, (m_1, m_{23}, m_4, m_5, m_{35})^\pm \,;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi^\pm_{f'} &= \{\, (m_{12}, m_3, m_4, m_5, m_{25})^\pm \,;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{f'''} &= \{\, (m_1, m_2, m_{35}, m_5, m_4)^\pm \,;\, \pm\tfrac{1}{2} m_{13} \,\} \\
\chi^\pm_{g} &= \{\, (m_2, m_3, m_4, m_5, m_{15})^\pm \,;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{g''} &= \{\, (m_1, m_{23}, m_{45}, m_5, m_{34})^\pm \,;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi^\pm_{h'} &= \{\, (m_{12}, m_3, m_{45}, m_5, m_{24})^\pm \,;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{h''} &= \{\, (m_2, m_3, m_{45}, m_5, m_{14})^\pm \,;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{k} &= \{\, (m_2, m_{34}, m_5, m_{45}, m_{13})^\pm \,;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{k'} &= \{\, (m_{12}, m_{34}, m_5, m_{45}, m_{23})^\pm \,;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{k''} &= \{\, (m_1, m_{24}, m_5, m_{45}, m_3)^\pm \,;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi^\pm_{\ell} &= \{\, (m_2, m_{35}, m_5, m_4, m_{13})^\pm \,;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{\ell'} &= \{\, (m_{12}, m_{35}, m_5, m_4, m_{23})^\pm \,;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi^\pm_{\ell''} &= \{\, (m_1, m_{25}, m_5, m_4, m_3)^\pm \,;\, \pm\tfrac{1}{2} m_{12} \,\}
\end{aligned}
$$

The multiplets are given explicitly in Fig. 1f.

Here the ER χ_0^+ contains the limits of the (anti)holomorphic discrete series representations. This is guaranteed by the fact that for this ER all Harish-Chandra parameters for non-compact roots are non-positive, i.e., $m_\alpha \leq 0$, for α from (16). (Actually, we have: $m_{11} = 0$, $m_\alpha < 0$ for the rest of the non-compact α.) Its conformal weight has the restriction $d = \frac{7}{2} + \frac{1}{2}(m_1 + \cdots + m_5) \geq 6$.

4.2 The Cases sp(n,$I\!R$) for $n \leq 5$

We start with $sp(5, I\!R)$. The main multiplets R^5 contain $32(= 2^5)$ ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \{\, (m_1, m_2, m_3, m_4)^\pm \,;\, \pm\tfrac{1}{2}(m_{14} + 2m_5) \,\} \\
\chi_a^\pm &= \{\, (m_1, m_2, m_3, m_4 + 2m_5)^\pm \,;\, \pm\tfrac{1}{2}m_{14} \,\} \\
\chi_b^\pm &= \{\, (m_1, m_2, m_{34}, m_4 + 2m_5)^\pm \,;\, \pm\tfrac{1}{2}m_{13} \,\} \\
\chi_c^\pm &= \{\, (m_1, m_{23}, m_4, m_{34} + 2m_5)^\pm \,;\, \pm\tfrac{1}{2}m_{12} \,\} \\
\chi_{c'}^\pm &= \{\, (m_1, m_2, m_{34} + 2m_5, m_4)^\pm \,;\, \pm\tfrac{1}{2}m_{13} \,\} \\
\chi_d^\pm &= \{\, (m_{12}, m_3, m_4, m_{24} + 2m_5)^\pm \,;\, \pm\tfrac{1}{2}m_1 \,\} \\
\chi_{d'}^\pm &= \{\, (m_1, m_{23}, m_4 + 2m_5, m_{34})^\pm \,;\, \pm\tfrac{1}{2}m_{12} \,\} \\
\chi_e^\pm &= \{\, (m_2, m_3, m_4, m_{14} + 2m_5)^\pm \,;\, \mp\tfrac{1}{2}m_1 \,\} \\
\chi_{e'}^\pm &= \{\, (m_{12}, m_3, m_4 + 2m_5, m_{24})^\pm \,;\, \pm\tfrac{1}{2}m_1 \,\} \\
\chi_{e''}^\pm &= \{\, (m_1, m_{24}, m_4 + 2m_5, m_3)^\pm \,;\, \pm\tfrac{1}{2}m_{12} \,\} \\
\chi_f^\pm &= \{\, (m_2, m_3, m_4 + 2m_5, m_{14})^\pm \,;\, \mp\tfrac{1}{2}m_1 \,\} \\
\chi_{f'}^\pm &= \{\, (m_{12}, m_{34}, m_4 + 2m_5, m_{23})^\pm \,;\, \pm\tfrac{1}{2}m_1 \,\} \\
\chi_{f''}^\pm &= \{\, (m_1, m_{24} + 2m_5, m_4, m_3)^\pm \,;\, \pm\tfrac{1}{2}m_{12} \,\} \\
\chi_g^\pm &= \{\, (m_2, m_{34}, m_4 + 2m_5, m_{13})^\pm \,;\, \mp\tfrac{1}{2}m_1 \,\} \\
\chi_{g'}^\pm &= \{\, (m_{12}, m_{34} + 2m_5, m_4, m_{23})^\pm \,;\, \pm\tfrac{1}{2}m_1 \,\} \\
\chi_h^\pm &= \{\, (m_2, m_{34} + 2m_5, m_4, m_{13})^\pm \,;\, \mp\tfrac{1}{2}m_1 \,\}
\end{aligned} \tag{27}$$

Recalling that the $Sp(6, I\!R)$ reduced multiplets of type R_6^6 also have 32 members we check whether they may be coinciding. Indeed, that turns out to be the case and this is obvious from the corresponding figures, Fig. 1f and Fig. 2 (though our graphical representations are a little distorted!). To make it explicit via the signatures we do the following manipulations of Table (26) : in each signature we just drop the entry m_5 (there is exactly one such entry in each signature). Then we replace each entry of the kind: m_{k5}, $(k = 1, 2, 3, 4)$, by $m_{k4} + 2m_5$ (identifying $m_{44} \equiv m_4$). Thus (26) becomes exactly (27). (Of course, this does not mean that the contents is the same.

For instance, the ER χ_0^+ from (27) contains the (anti)holomorphic discrete series representations of $sp(5,\mathbb{R})$, while the ER χ_0^+ from (26) contains the limits of the (anti)holomorphic discrete series representations of $sp(6,\mathbb{R})$.)

Thus, it is clear how to obtain from the case $sp(6,\mathbb{R})$ all the cases $sp(n,\mathbb{R})$ for $n \leq 5$. We shall not do it here due to the lack of space.

5 Outlook

In the present paper we continued the programme outlined in [2] on the example of the non-compact group $Sp(n,\mathbb{R})$. Similar explicit descriptions are planned for the other non-compact groups, in particular, those with highest/lowest weight representations. From the latter we have considered so far the cases of $E_{7(-25)}$ [3][4], $E_{6(-14)}$ [29], $SU(n,n)$ $(n \leq 4)$ [30]. We plan also to extend these considerations to the supersymmetric cases and also to the quantum group setting. Such considerations are expected to be very useful for applications to string theory and integrable models, cf., e.g., [31]. In our further plans it shall be very useful that (as in [2]) we follow a procedure in representation theory in which intertwining differential operators appear canonically [7] and which procedure has been generalized to the supersymmetry setting [32, 33] and to quantum groups [34].

Acknowledgments. This work was supported in part by the Bulgarian National Science Fund, grant DO 02-257. The author thanks the Theory Division of CERN for hospitality during the course of this work.

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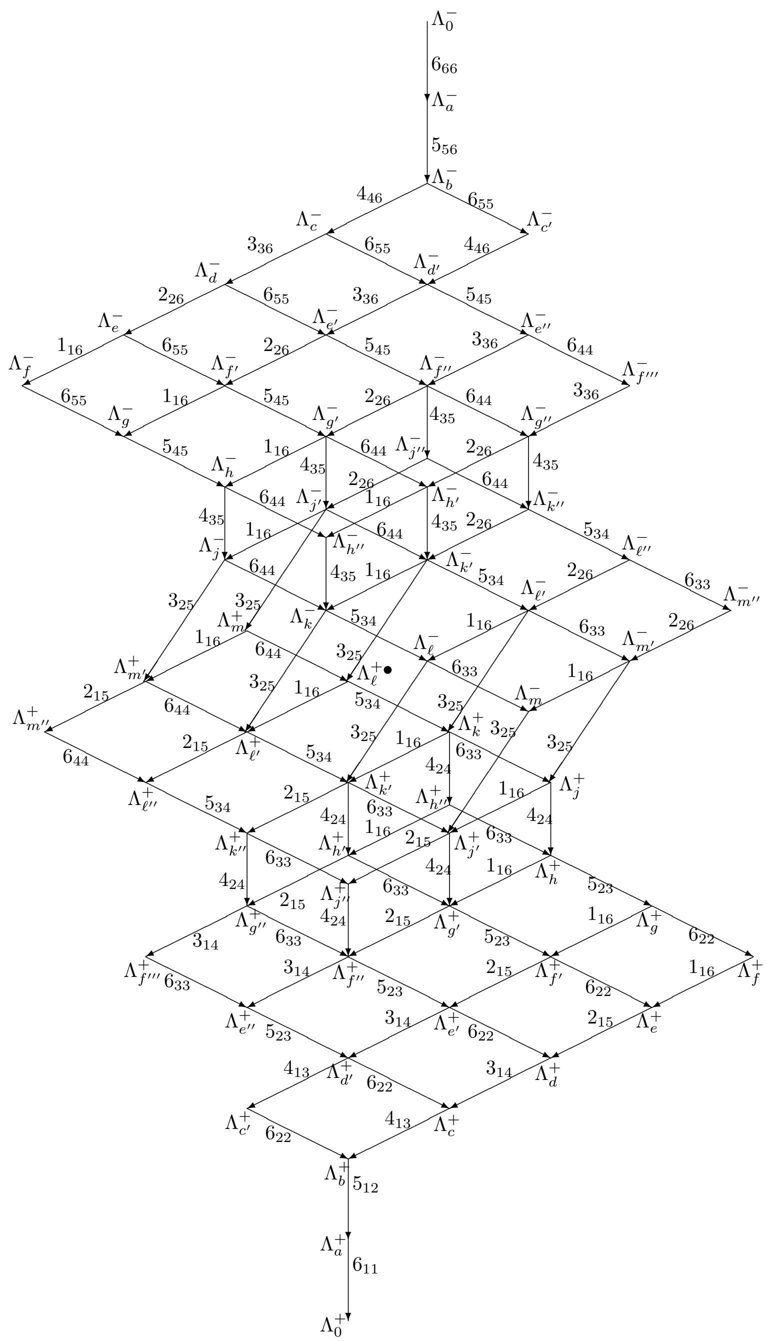

Fig. 1. Main multiplets for $Sp(6,I\!R)$

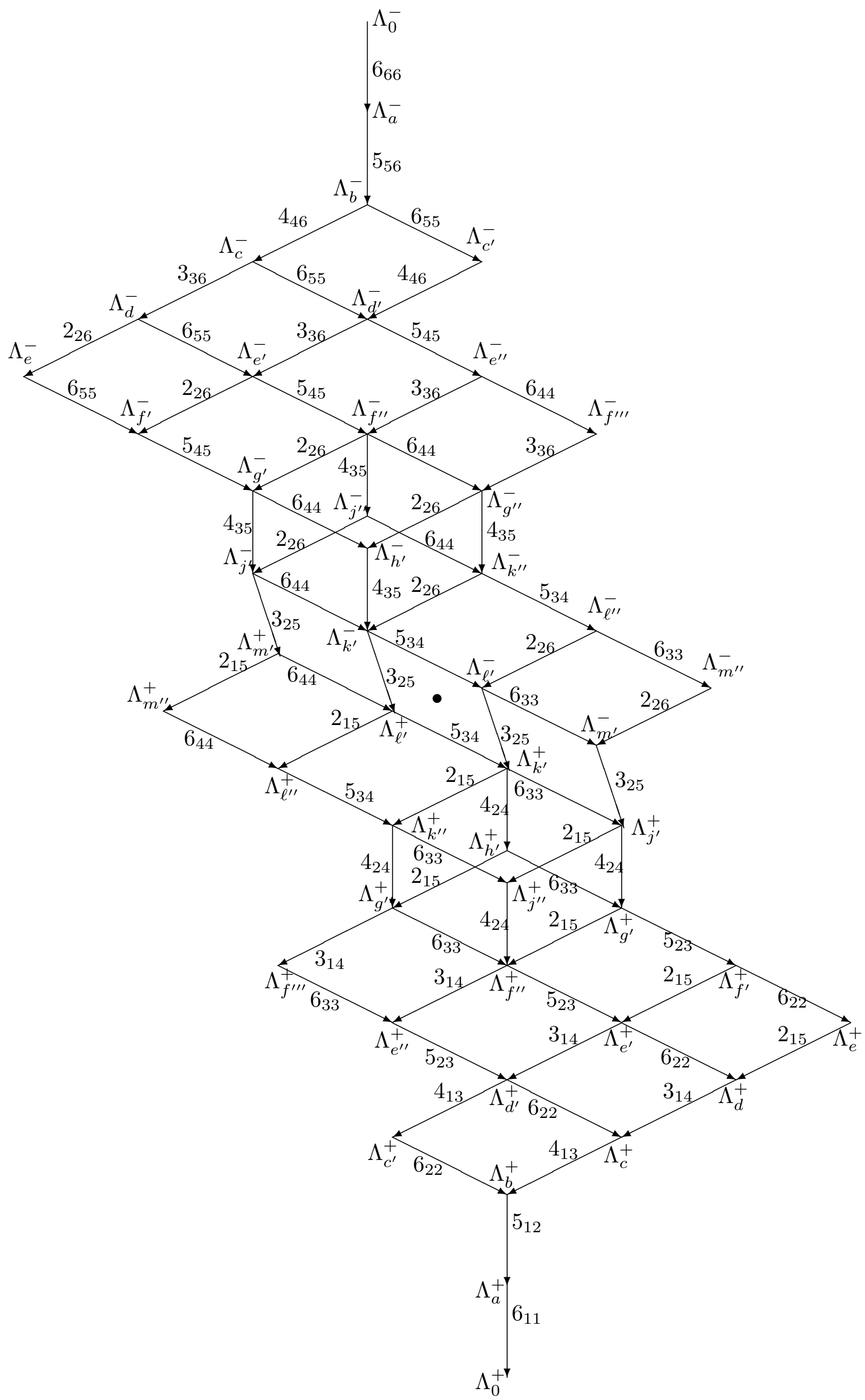

Fig. 1a. Reduced multiplets R_1^6 for $Sp(6,I\!R)$

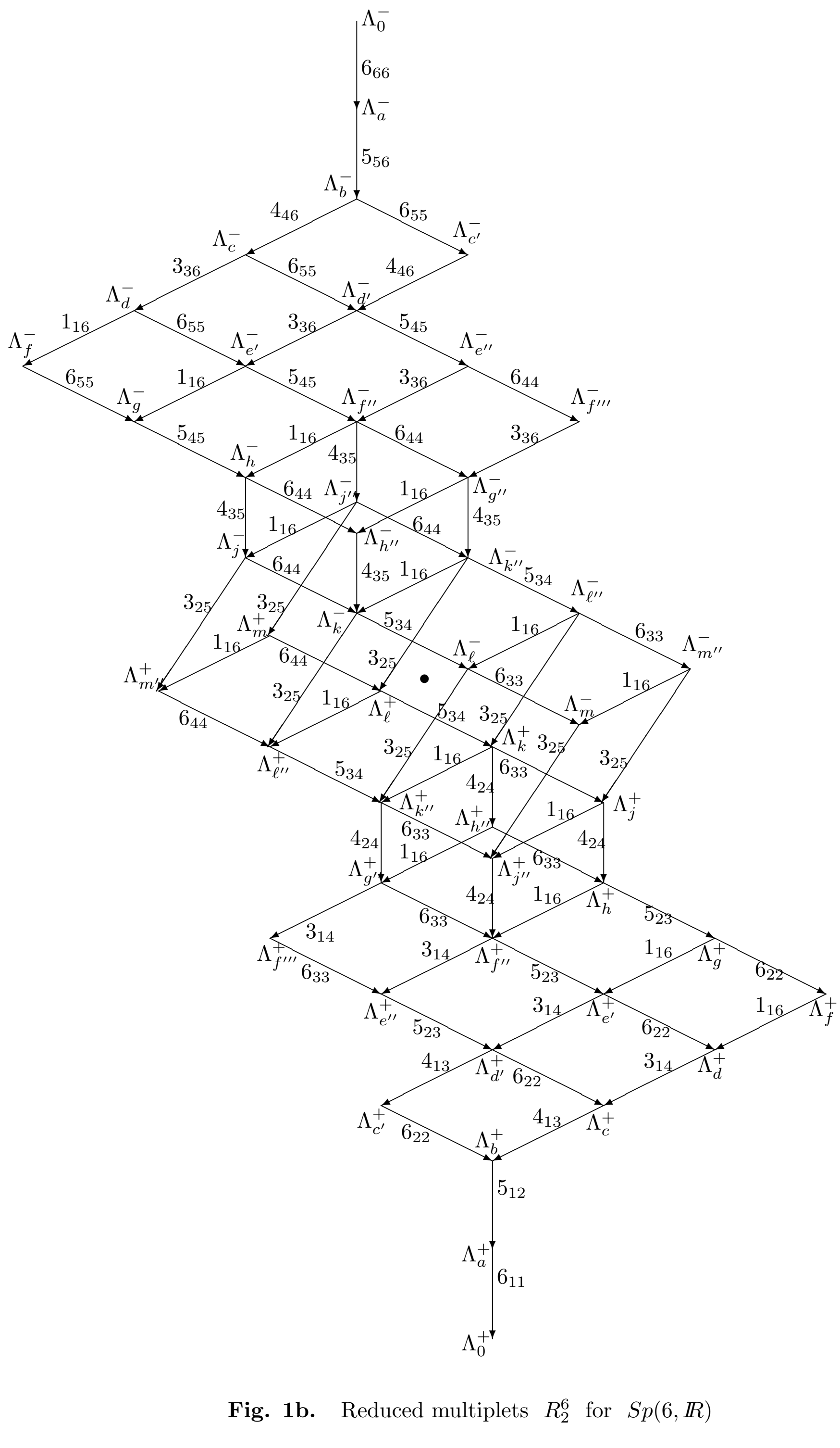

Fig. 1b. Reduced multiplets R_2^6 for $Sp(6,I\!R)$

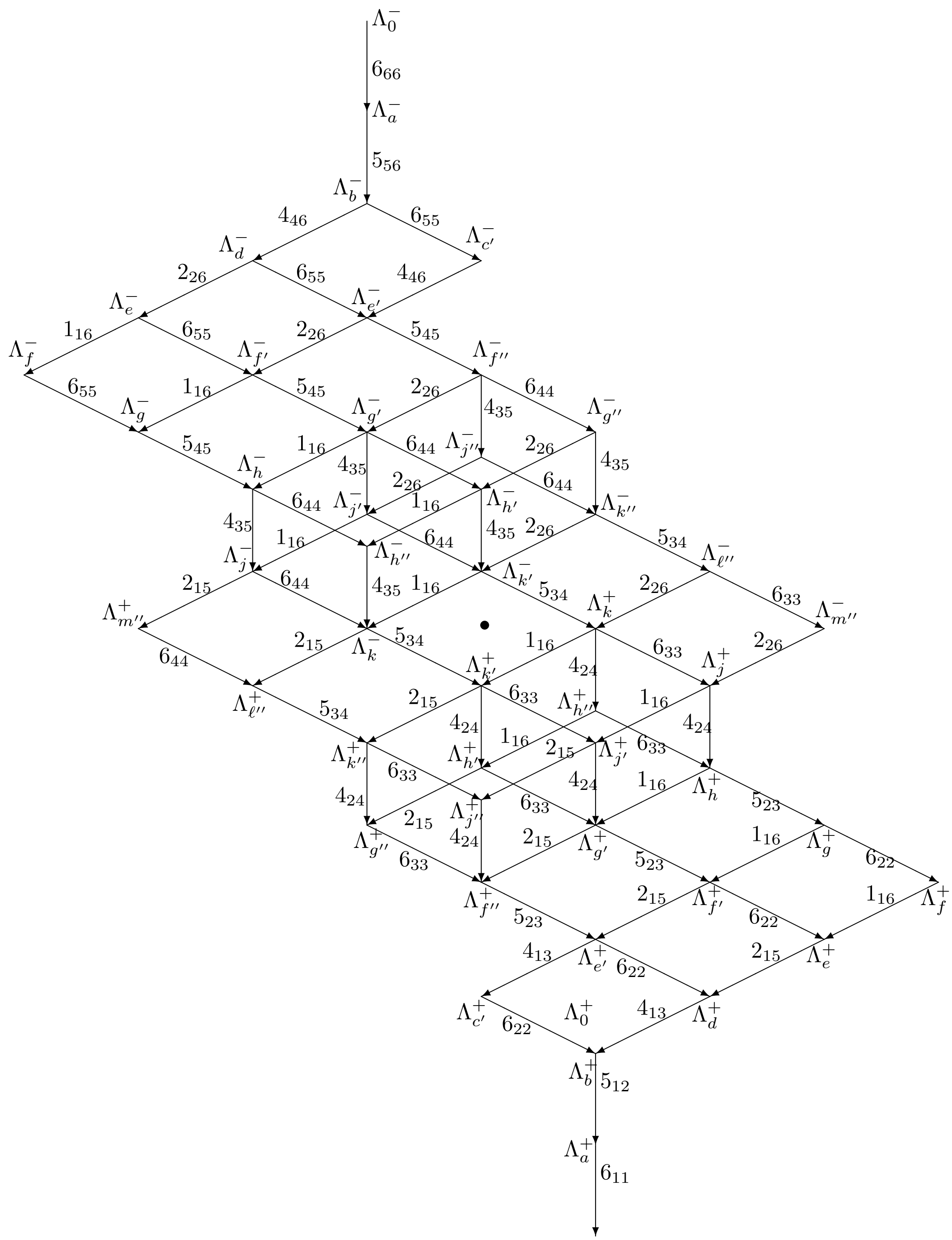

Fig. 1c. Reduced multiplets R_3^6 for $Sp(6,\mathbb{R})$

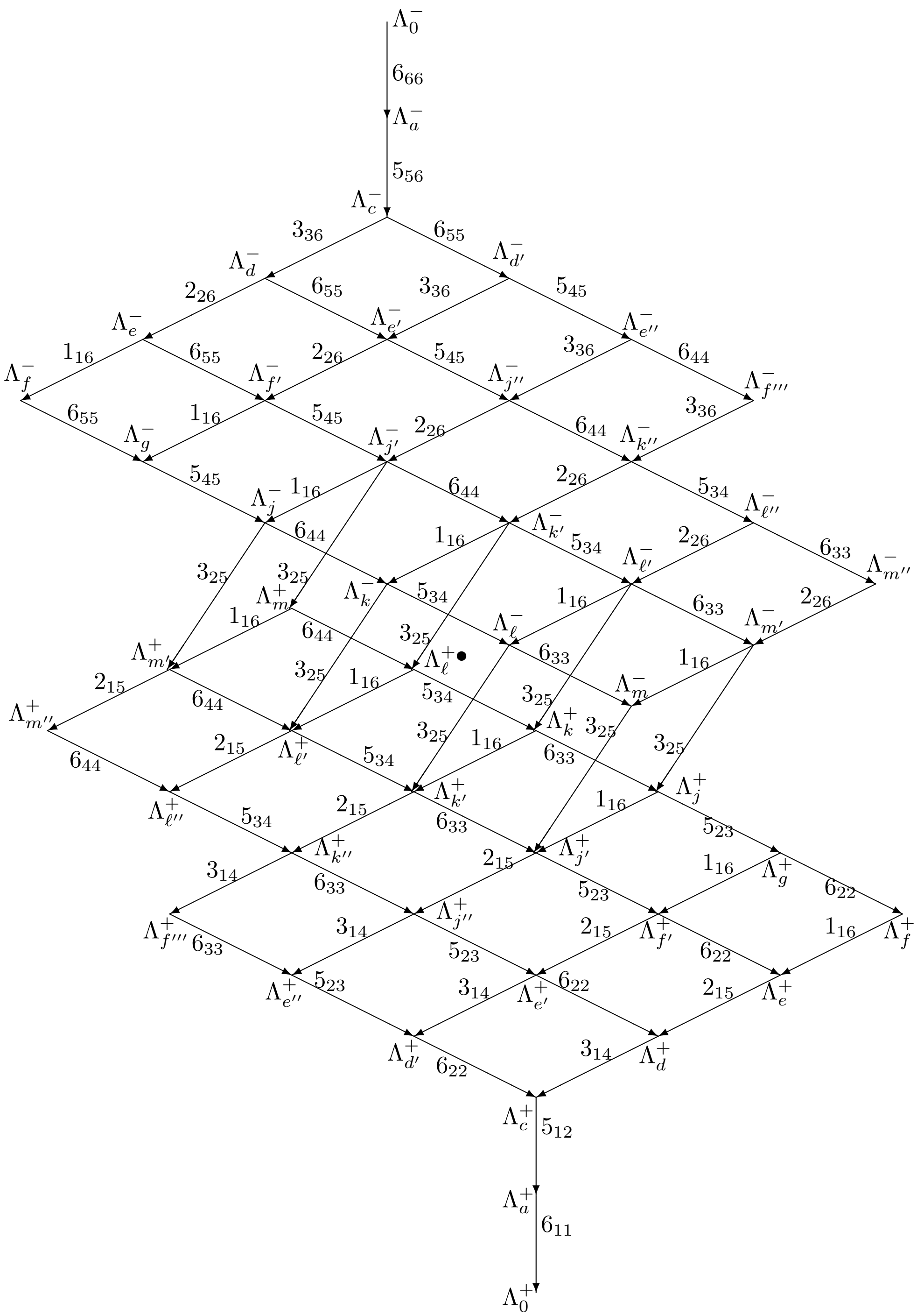

Fig. 1d. Reduced multiplets R_4^6 for $Sp(6,I\!R)$

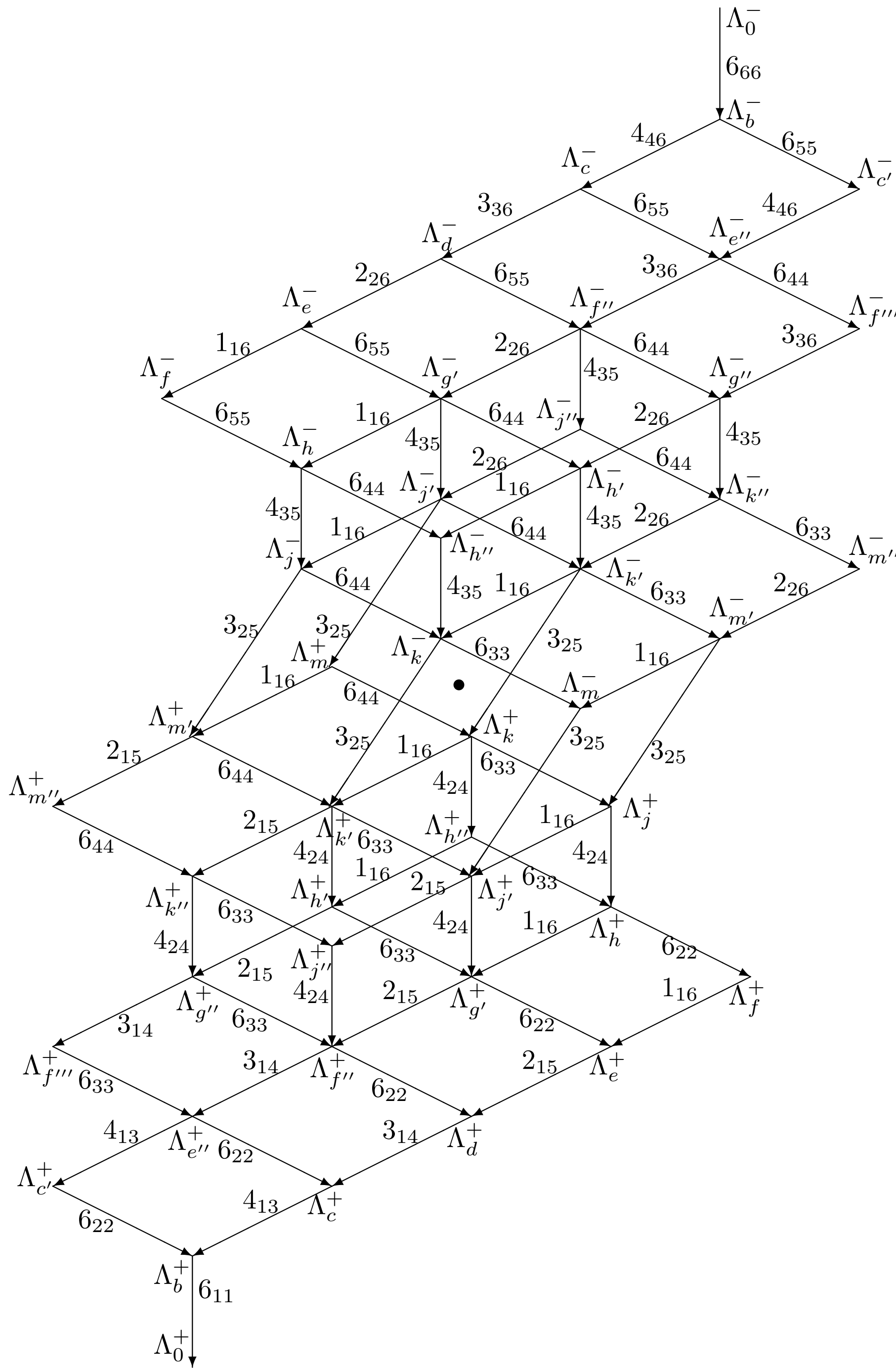

Fig. 1e. Reduced multiplets R_5^6 for $Sp(6,I\!R)$

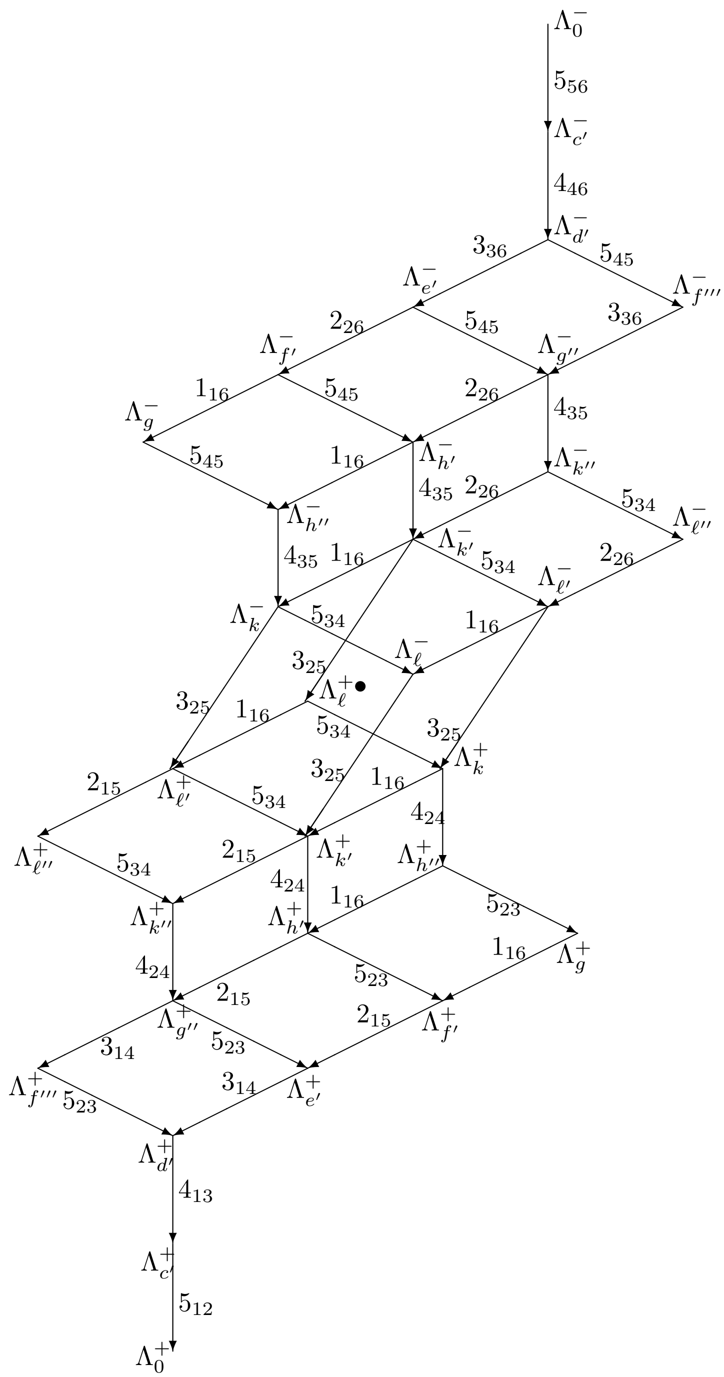

Fig. 1f. Reduced multiplets R_6^6 for $Sp(6,\mathbb{R})$

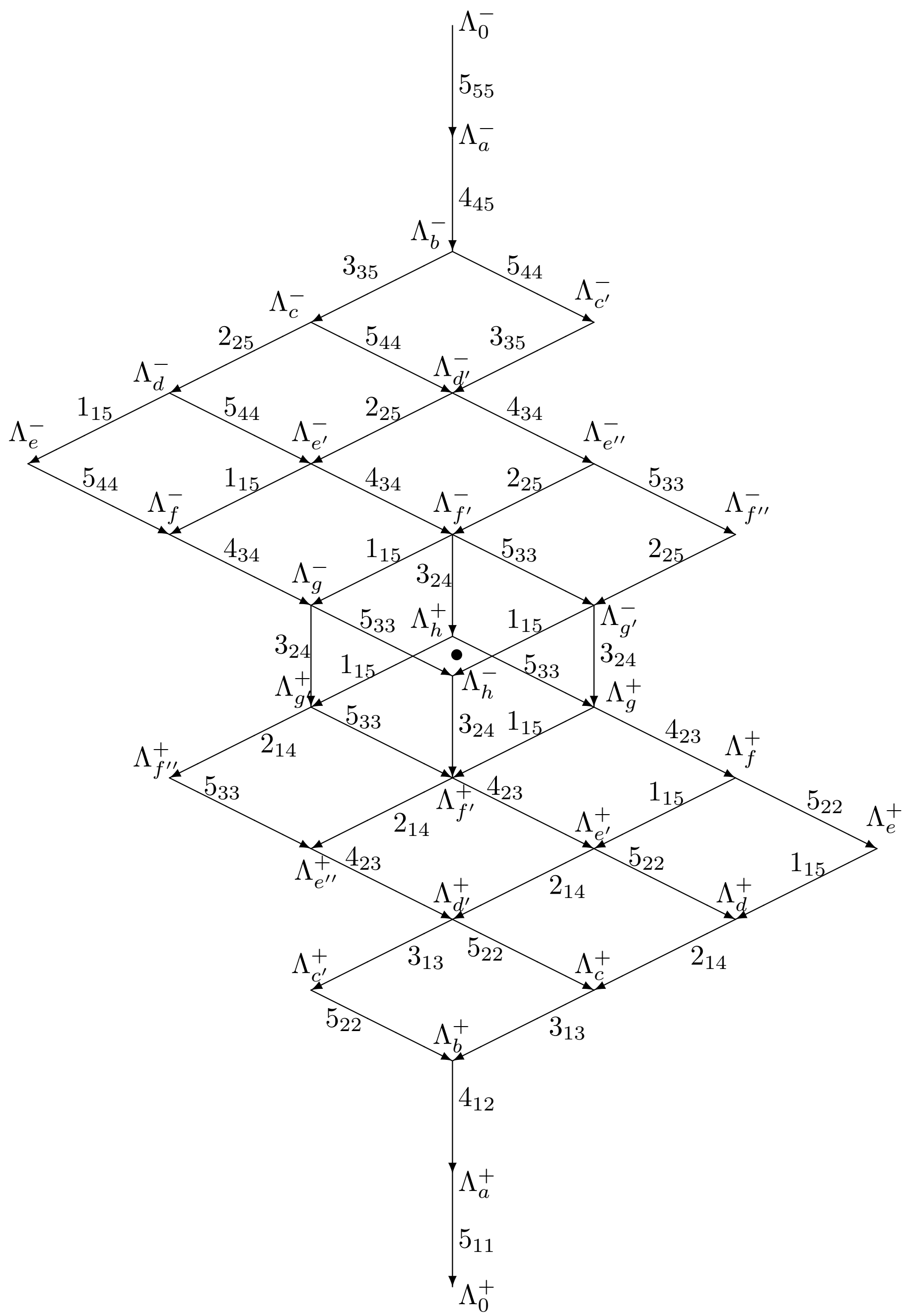

Fig. 2. Main multiplets R^5 for $Sp(5,I\!R)$